Interpretation of the dynamic Allan variance of nonstationary clock data

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***Abstract*— Atomic clocks are ultra-stable time references, and they have found fundamental applications in several fields, navigation being certainly one of the most successful. Unfortunately, atomic clocks may experience nonstationary behaviors which make their stability a function of time. Even when the change in stability is thought to be small, it can still force the system that uses the clock to fail the design requirements. In navigation systems this might turn into an unwanted degradation of the user localization error. It is therefore essential to monitor the stability of the clock, and to understand and interpret the possible nonstationary behaviors. To this aim, we have recently introduced the dynamic Allan variance, or DAVAR, a quantity that allows to represent the evolution with time of the clock stability. In this paper we show several cases of nonstationary time series, made by the combination of random processes and deterministic signals, and we discuss their dynamic Allan variance. The ultimate goal is to be able to do the inverse operation, that is to understand and classify the possible nonstationarities directly from the DAVAR representation. Examples with experimental data are presented.**

I. Introduction

The reason why atomic clocks have found many applications is due to their extremely high stability. Navigation is probably the most famous field where atomic clocks are used with dramatic effects. By using atomic clocks the Galileo system currently under development aims at localizing the user with an error of few tens of centimeters. Unfortunately a slight variation in the clock stability can turn into an increased error in the user localization, even up to several meters. It is therefore fundamental to monitor the clock stability as a function of time. We have recently introduce the dynamic Allan variance, or DAVAR, a representation of the time-varying stability of an atomic clock. By using the DAVAR one can recognize the possible nonstationarities that can happen in clock, and that are mainly due to sudden failures, aging and variations in the temperature [3], [2], [8], [6].

We are presently working at the classification of all the possible nonstationary behaviors of an atomic clock. Our aim is to be able to detect and recognize them directly from the DAVAR surface. This is particularly interesting since many of them cannot be spotted from the time signal or the Allan variance alone. In this paper we review the main ideas behind the dynamic Allan variance and we characterize some types of nonstationary behaviors. We also show that these nonstationarities can be found in real clocks.

We point that the dynamic Allan variance has recently received some attention, and in particular:

- it has been implemented in the STABLE32 software (`www.wriley.com`);
- it has been implemented in the CANVAS software, developed by the US NAVY (`https://goby.nrl.navy.mil/canvas/`)
- it is used in the characterization of the clocks onboard the first Galileo experimental satellite, GIOVE-A [8];
- it has been proposed as a tool for robust deep space time keeping [5].

II. The Allan variance

If $\boldsymbol{y}(t)$ is the fractional frequency deviation [4] of the atomic clock error (the deviation from an ideal reference), then the phase deviation can be defined as[1]

$$\boldsymbol{y}(t) = \frac{d\boldsymbol{x}(t)}{dt} \tag{1}$$

The quantities $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are random processes, and in atomic clocks they have, in general, a frequency spectrum with a power-law behavior, that is $P_{\boldsymbol{x}}(f) = 1/f^{\alpha}$. The Allan variance[1] is a measure of the variability in time of the frequency $\boldsymbol{y}(t)$ over different observation intervals τ. The 2-sample Allan variance is the most used form

$$\sigma^2_{\boldsymbol{y}}(\tau) = \frac{1}{2}\left\langle \left(\bar{\boldsymbol{y}}_{t+\tau} - \bar{\boldsymbol{y}}_t\right)^2 \right\rangle \tag{2}$$

where τ is the observation interval, the operator $\langle\cdot\rangle$ stands for time averaging and

$$\bar{\boldsymbol{y}}(t) = \frac{1}{\tau}\int_{t-\tau}^{t} \boldsymbol{y}(t)dt = \frac{\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau)}{\tau} \tag{3}$$

The Allan variance is usually represented in bi-logarithmic scale, so that the spectra of the typical clock noises are represented by straight lines, where the slope of the line can be used to classify the type of noise.

We can estimate the Allan variance from discrete time phase data $\boldsymbol{x}[n] = \boldsymbol{x}(n\tau_0)$, where τ_0 is the sampling time, by using

$$\boldsymbol{\sigma}^2_{\boldsymbol{y}}[k] = \frac{1}{2k^2\tau_0^2}\frac{1}{N-2k}\times \sum_{n=0}^{N-2k-1} \left(\boldsymbol{x}[n+2k] - 2\boldsymbol{x}[n+k] + \boldsymbol{x}[n]\right)^2 \tag{4}$$

[1]We use bold letters for random quantities.

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The quantity $k = \tau/\tau_0$ is the discrete time observation interval. The square root of the Allan variance is the Allan deviation $\boldsymbol{\sigma_y}[k]$.

III. The dynamic Allan variance

The dynamic Allan variance is a natural extension of the Allan variance, and it is obtained by sliding the Allan variance on the data. At a given time t_1 we truncate the signal with a rectangular window centered about t_1 and we discard everything that is outside the window. Then we compute the Allan variance $\sigma^2_{\boldsymbol{y}}(t_1, \tau)$.of the truncated signal. We repeat the same procedure for all times, and the collection of all the obtained Allan variances represents the dynamic Allan variance $\sigma^2_{\boldsymbol{y}}(t, \tau)$. The dynamic Allan variance is a three-dimensional surface that represents the variation with time of the clock stability.

By following the intuitive definition one can derive a mathematical formulation of the dynamic Allan variance. Starting from the discrete time phase signal $x[n]$, we define the DAVAR as

$$\sigma^2_{\boldsymbol{y}}[n,k] = \frac{1}{2k^2\tau_0^2}\frac{1}{N-2k}\times \sum_{m=n-N/2+k}^{n+N/2-k-1} E\left[(\boldsymbol{x}[m+k] - 2\boldsymbol{x}[m] + \boldsymbol{x}[m-k])^2\right] \quad (5)$$

where $n = t/\tau_0$ is the discrete time, $k = \tau/\tau_0$ is the observation interval in discrete time, and N is the length of the window. It is

$$k = 0, 1, \ldots, \frac{N}{2} - 1 \quad (6)$$

A similar expression can be derived from the frequency data $\boldsymbol{y}[n]$. The dynamic Allan deviation $\sigma_{\boldsymbol{y}}[n,k]$, or DADEV, is the square root of the dynamic Allan variance.

The definition requires the expectation value because we want $\sigma^2_{\boldsymbol{y}}[n,k]$ to be a deterministic quantity. In this way we can characterize the time-varying stability of a clock noise without considering the random fluctuations that would arise in the DAVAR surface without ensemble averaging. The dynamic Allan variance can be estimated from experimental data by using

$$\hat{\sigma}^2_{\boldsymbol{y}}[n,k] = \frac{1}{2k^2\tau_0^2}\frac{1}{N-2k}\times \sum_{m=n-N/2+k}^{n+N/2-k-1} (\boldsymbol{x}[m+k] - 2\boldsymbol{x}[m] + \boldsymbol{x}[m-k])^2 \quad (7)$$

Where the stochastic quantity $\hat{\sigma}^2_{\boldsymbol{y}}[n,k]$ is therefore the estimator of the dynamic Allan variance.

IV. Nonstationary clock analysis: case 1

We now review some nonstationary behaviors that can happen in atomic clocks. First we perform some simulations, and then we show how the obtained results can be used to interpret experimental data coming from a Rubidium clock.

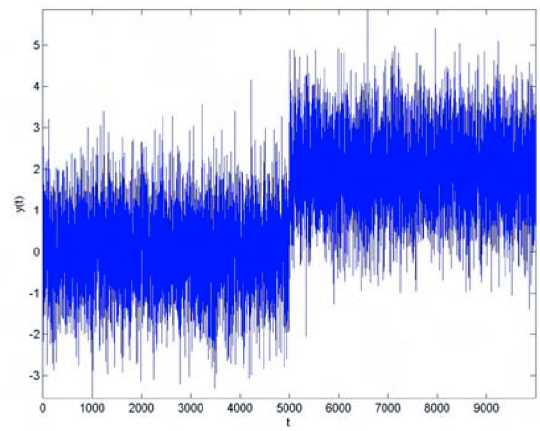

Fig. 1. White frequency noise with a change in the mean.

A. Change in the mean

Suppose that we have a white frequency noise with a given mean and variance. Suddenly, the mean increases while the variance remains constant. In Fig. 1 we show a possible realization of such noise, where the change in the variance happens in the middle of the signal. What is the dynamic Allan deviation of such signal? We have computed it and we show it in Fig. 2. It can be seen that before and after the nonstationary event the DADEV is stationary, indicating the presence of a white frequency noise. In the region where the event happens the DADEV is still flat for small observation intervals τ, but increases for large τ. The reason is that for small values of τ only a few triplets are affected by the change in the mean. By triplet we mean the three values $\boldsymbol{x}[m+k]$, $\boldsymbol{x}[m]$, and $\boldsymbol{x}[m-k]$ used in the estimation of the DAVAR, Eq. (7). Since the change in the mean is comparable with the fluctuations of the white noise, then the event is averaged out and cannot be seen in the resulting DADEV surface. On the contrary, when τ is large, many triplets will have some of their values located before and after the change in the mean, and therefore the stability at that time will be higher.

B. Spike

In Fig. 3 we show a white frequency noise plus a spike in the middle. The spike is the numeric version of a delta function, a signal that is very often found in experimental signals, especially due to outliers. The corresponding dynamic Allan deviation is shown in Fig. 4. We see that the DADEV is stationary before and after the event. At the time when the event takes the place, we see that the stability increases for all τ values. This happens because the value of the delta function is very large with respect to the random fluctuations of the noise, and when a triplet tracks it its value is so different from the others that affects the entire stability for that τ.

C. Change in the variance

Another common nonstationary behavior is the slow variation in one if the clock parameters. In Fig. 5 we show a white

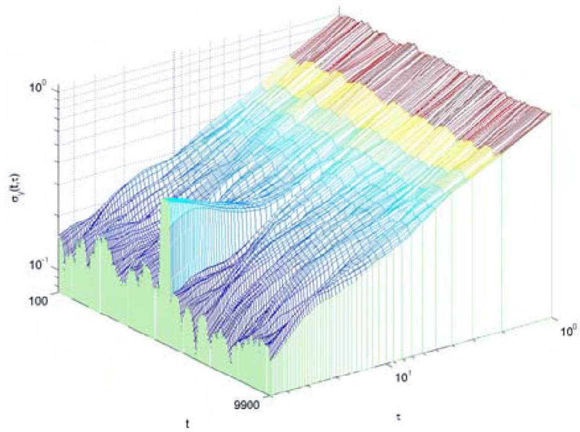

Fig. 2. Dynamic Allan deviation of the nonstationary signal shown in Fig. 1.

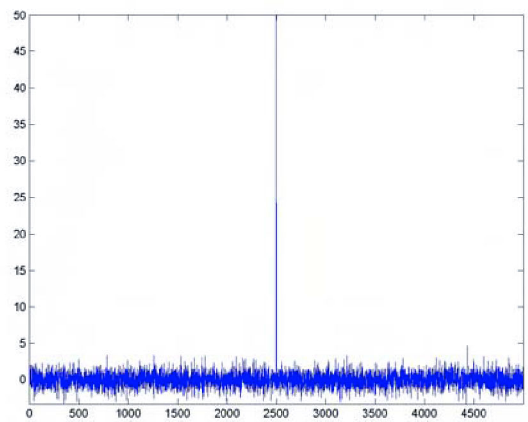

Fig. 3. White frequency noise plus a delta function.

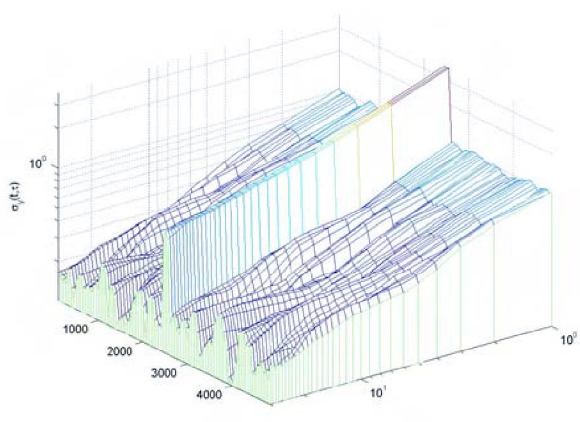

Fig. 4. Dynamic Allan deviation of the signal shown in Fig. 3.

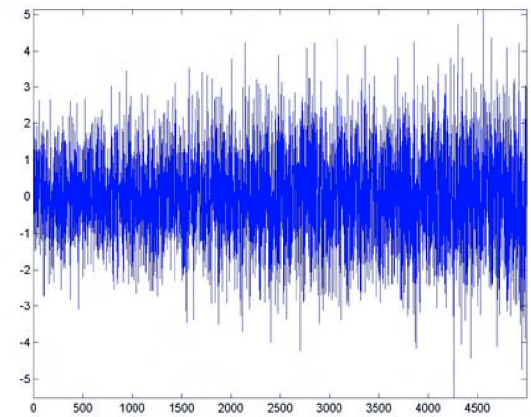

Fig. 5. White frequency noise whose standard deviaton increases linearly with time.

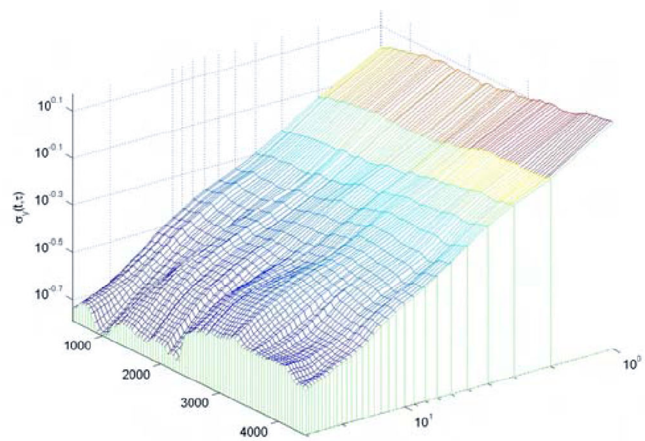

Fig. 6. Dynamic Allan deviation of the signal shown in Fig. 5.

frequency noise whose standard deviation increases linearly. In Fig. 6 we report the corresponding dynamic Allan deviation. It can be seen, both for small and large τ values, the slowly varying increase in the DADEV surface.

D. Experimental data

In Fig. 7 we show the frequency of a Rubidium clock that is being tested for space flight certification. We see that the signal is nonstationary, and we immediately spot some delta functions (outliers) and several changes in the mean of the signal. The Allan deviation, given in Fig. 8, unfortunately does not allow to understand the nonstationary events going on in the signal. In Fig. 9 we show the corresponding dynamic Allan deviation. Three things can be noticed:

1) There are "tails" that come out from the DADEV surface at large τ values. These are generated by the frequent changes in the mean of the signal. They do not diverge

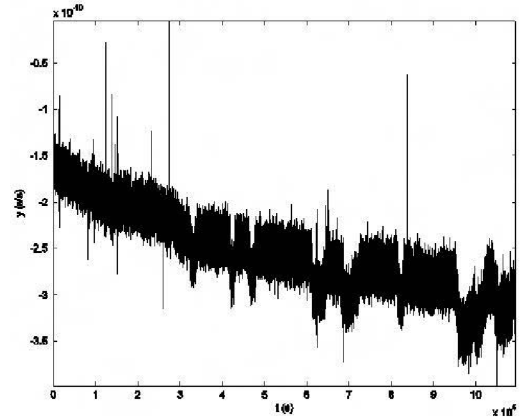

Fig. 7. Frequency deviation of a Rubidium clock undergoing tests for space flight certification.

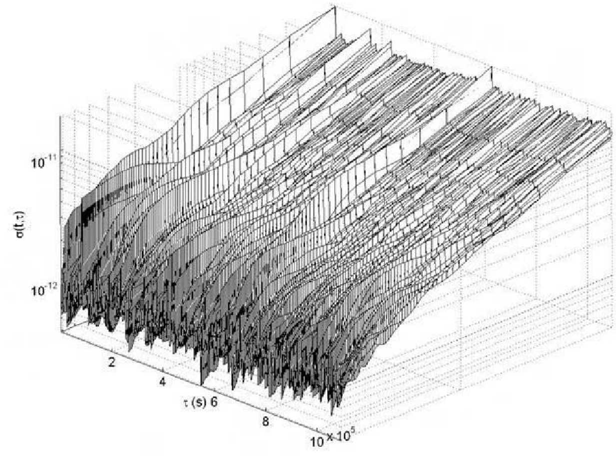

Fig. 9. Dynamic Allan deviation of the signal shown in Fig. 7.

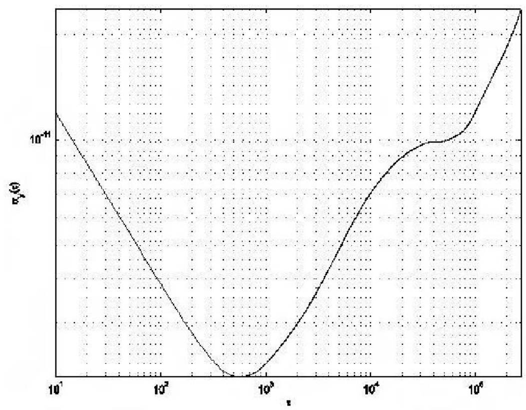

Fig. 8. Allan deviation of the signal shown in Fig. 7.

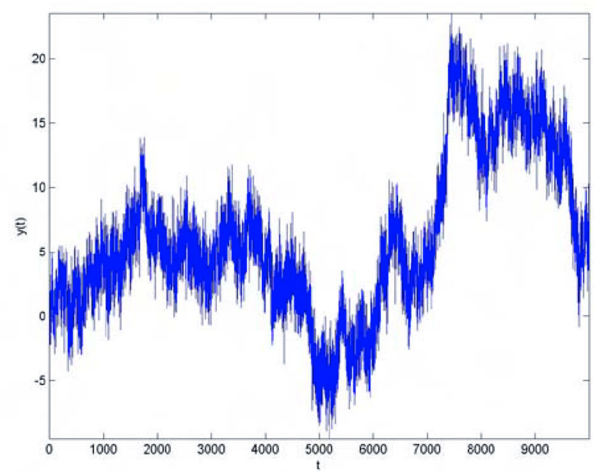

Fig. 10. Frequency deviation of a simulated clock whose model has a time-varying parameter.

as in the simulated case, Fig. 2, because the variations in the mean do not persist for all times, but they are rather followed by other variations.

2) There are some abrupt changes in the stability localized at given times and for all τ values. These bumps are due to the delta functions visible in the time signal.
3) There is a slight increase in the variance that can be noticed especially for small τ values. This variation is very hard to be recognized in the time signal.

Therefore we conclude that the simulations carried on are very useful to understand the nonstationary nature of the Rubidium signal. Also, knowing their signature in the DADEV domain allows one to directly identify the type of nonstationarity going on, potentially without looking at the time signal.

V. Nonstationary clock analysis: case 2

A nonstationary behavior can happen in general because of a variation of a clock parameter. In what follows we will show how such variation can be used to explain a nonstationary behavior in a GPS clock.

A. Change in the model parameter

Suppose that we model an atomic clock by adding a white frequency noise and a random walk frequency noise whose standard deviation increases linearly with time. In Fig. 3 we show a simulated realization of this signal. It is very hard to detect the variation in the model that we know is going on in the signal. To do that, we need to look at the dynamic Allan deviation, represented in Fig. 4. We see that for large τ values the increase in the variance is clearly visible.

B. Experimental data

In Fig. 12 we show the frequency signal of a GPS clock. It is very hard to understand from the time signal whether there are nonstationary behavior going on in the clock. In Fig. 13 we show the corresponding dynamic Allan deviation. We

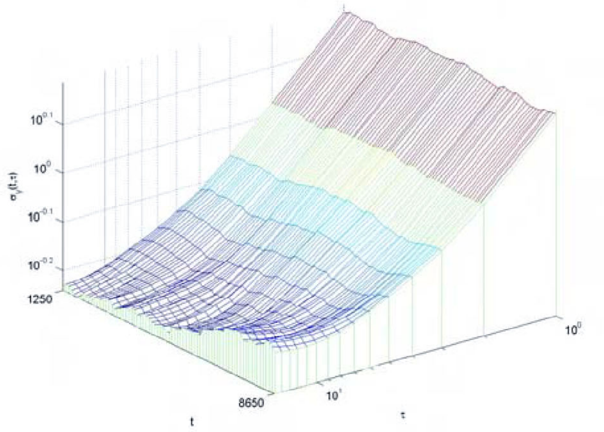

Fig. 11. Dynamic Allan deviation of the signal shown in Fig. 4

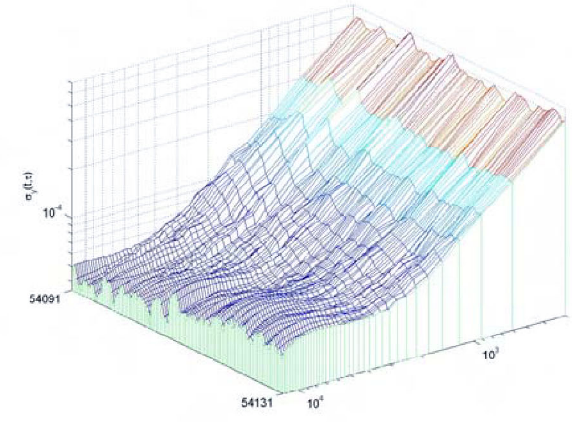

Fig. 13. Dynamic Allan deviation of the signal shown in Fig. 12.

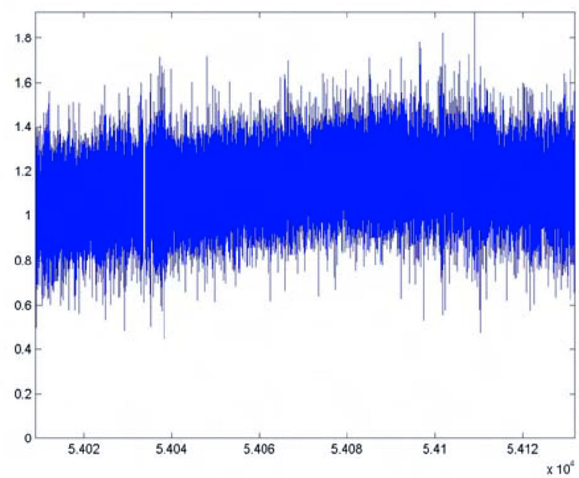

Fig. 12. Frequency deviation of a GPS clock.

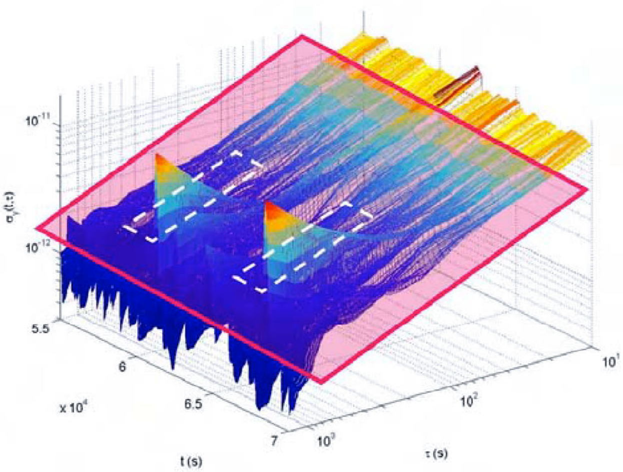

Fig. 14. Dynamic Allan deviation of experimental data measured from a Rubidium clock. The red plane is a reference surface for anomaly detection.

clearly see that there is an increase in the DADEV surface for large τ values, while the surface is pretty stationary for small τ. This nonstationarity could be explained by using a time-varying model similar to the one proposed in Sect. V-A.

C. Applications

The dynamic Allan variance can be applied in several different ways. In general one can think of two ways of using the DAVAR:

1) To prove that a clock is behaving in a nonstationary way and to identify the types of nonstationarities present by studying the DAVAR surface.
2) To guarantee that a clock is behaving in a stationary way. For example a clock manufacturer might be interested in showing to the customer the DAVAR of an atomic clock, to prove that it maintains its stability for all times in a given period. A clock may in fact fail the requirements locally but meet the global requirements specified by the classical Allan variance. This "violation" of the requirements can be revealed only by showing the dynamic Allan variance.

In addition to this the DAVAR can be used to design fault detection algorithms [7]. One can define the requirements by giving a threshold surface $\sigma_{\mathrm{TH}}(t,\tau)$. An alarm will be raised every time that $\hat{\sigma}_{\boldsymbol{y}}(t,\tau) > \sigma_{\mathrm{TH}}(t,\tau)$. In Fig. 14 we show the dynamic Allan deviation of a Rubidium clock. The reference surface is the red plane. The two events highlighted by the dashed boxes exceed the given threshold.

VI. Conclusion

The dynamic Allan variance can be used to represent the variations in time of the stability of an atomic clock. The change in the clock stability can be due to many types of nonstationarities that can happen in the clock. It is fundamental to classify all the possible nonstationary behaviors and to characterize their corresponding dynamic Allan variance, so that by direct inspection of the DAVAR surface one can ascertain what type of events are going on in the clock. This is especially useful for those events that cannot be seen from the time signal alone. In this paper we have characterized some nonstationary behaviors and we have used the obtained results to explain the nonstationary nature of two real clocks. A free Matlab implementation of the DAVAR can be found at `www.ien.it/tf/ts/clock_behavior.shtml`.

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